Grade 6 Math Circles
Oct 11/12/13, 2022
Counting - Problem Set Solutions

1. A palindrome is a whole number which is the same whether you read it normally (left to right) or backwards (right to left). For example 6, 22, 313, 4884, 54745, and 612216 are all palindromes. How many 5-digit palindromes are there?

    Solution: Any 5-digit palindrome can be uniquely determined by its first 3 digits. That is, a 5-digit number is a palindrome if its first digit is its last digit, and its second digit is its second last digit. Therefore, the number of 5-digit palindromes is equal to the number of 3-digit whole numbers: $9 \times 10 \times 10 = 900$.

2. At the Gauss School of Mathematics, there are 17 members of the math team. Each of these members is taking at least one foreign language class. The school only offers French and German as its foreign language classes. 13 members of the math team members are taking French and 4 are taking both French and German. How many math team members are taking German?

    Solution: Since there are 13 members of the math team taking French, there are 4 members of the math team not taking French. All of these must be taking German. Therefore there are 4 members taking German but not French, and 4 members taking German and French. Hence, there are $4 + 4 = 8$ members of the math team taking German.

3. You have 2 pairs of shoes, 3 pairs of socks, 5 pairs of pants, and 9 shirts. An outfit consists of one pair of shoes, one pair of socks, one pair of pants, and one shirt. How many outfits can you make?

    Solution: Using the Fundamental Counting Principle, there are $2 \times 3 \times 5 \times 9 = 270$ possible outfits you can make.

4. Evaluate the following:

    (i) $\frac{19!-18!}{17!}$

(ii) ${}_{178}P_1 \ \times \ {}_{100}P_1$

(iii) ${}_{9}C_4 \ + \ {}_{9}C_5 \ - \ {}_{10}C_5$

(iv) ${}_{10}C_5 \ + \ {}_{10}C_6 \ - \ {}_{11}C_6$

Solution:

(a) $\frac{19!-18!}{17!} = \frac{19!}{17!} - \frac{18!}{17!} = \frac{19\times18\times17!}{17!} - \frac{18\times17!}{17!} = 19 \times 18 - 18 = 324$

(b) For any n, ${}_{n}P_1 = \frac{n!}{(n-1)!} = n$ (since the number of ways to order 1 item from n items is n). So

$$ {}_{178}P_1 \ \times \ {}_{100}P_1 = 178 \times 100 = 17\ 800 $$

(c)

$$ {}_{9}C_4 \ + \ {}_{9}C_5 \ - \ {}_{10}C_5 = \frac{9!}{5! \times 4!} + \frac{9!}{5! \times 4!} - \frac{10!}{5! \times 5!} $$

$$ = \frac{2 \times 9!}{5! \times 4!} - \frac{10 \times 9!}{5 \times 5! \times 4!} $$

$$ = \frac{2 \times 9!}{5! \times 4!} - \frac{2 \times 9!}{5! \times 4!} $$

$$ = 0 $$

(d) We can calculate ${}_{9}C_4$, ${}_{9}C_5$, and ${}_{10}C_5$ either by hand (simplifying fractions) or with a calculator to find

$$ {}_{10}C_5 \ + \ {}_{10}C_6 \ - \ {}_{11}C_6 = \frac{10!}{5! \times 5!} + \frac{10!}{6! \times 4!} - \frac{11!}{6! \times 5!} $$

$$ = 252 + 210 - 462 = 0 $$

Look at the answers of part c) and d). Can you make a hypothesis about the value of ${}_{n}C_k \ + \ {}_{n}C_{k+1} \ - \ {}_{n+1}C_k$ for any $0 \leq k \leq n$?

5. The Friday Math Circles Grade 7/8 class has four grade 7 students and three grade 8 students. How many ways can they be seated in a row of 7 chairs such that at least two grade 7 students are sitting next to each other?

Solution:
We will use complementary counting. To avoid two grade 7s sitting beside each other, the seating must look like 7878787 (where 7 denotes a grade 7 student and 8 denotes a grade 8 student). If we order the grade 7s and the grade 8s separately, there will be exactly one such seating which maintains the orderings of the grade 7s and the grade 8s.

There are 4! ways to order the grade 7s and 3! ways to order the grade 8s therefore there are $4! \times 3! = 144$ seatings in which no two grade 7 students sit beside each other.

There are 7! seatings in total therefore there are $7! - 144 = 4896$ seatings such that at least two grade 7 students sit beside each other.

6. For each of the following words, how many ways can we rearrange the letters in the word? (eg. in Example 3, we saw that there are 24 ways of rearranging the letters of the word MATH)

(a) COUNT
(b) CIRCLES
(c) ONONONONONO

Solution:
(a) Each letter is distinct and there are 5 letters so there are $5! = 120$ rearrangements of the letters in COUNT.
(b) If we do the same thing as above, we find $7! = 5040$ rearrangements. However, the two C's are the same so we have counted every rearrangement twice. Hence, there are $\frac{5040}{2} = 2520$ rearrangements in total.
(c) There are 6 O's and 5 N's. Given 11 spots for the letters, there are ${}_{11}C_6$ ways of picking the 6 spots for the O's (and the N's will go in the remaining 5 spots). Therefore there are ${}_{11}C_6 = 462$ rearrangements in total.

7. How many ways can 5 students be seated at a round table? Two seatings are considered the same if, for every student, the person sitting on their right is the same in both seatings (i.e. can be achieved by rotation).

> *Solution*: Let A be the first student and fix A's place at the top of the round table (we can always 'spin' the table to put A at the top). There are now 4 fixed seats and 4 students which need a seat. This is just the number of rearrangements of 4 students. Therefore there are $4! = 24$ ways of seating the 5 students.

8. (*Gauss 2019 Gr. 8 # 21*) In Jen's baseball league, each team plays exactly 6 games against each of the other teams in the league. If a total of 396 games are played, how many teams are in the league?

> *Solution*: Let there be n teams. Each team plays $6 \times (n-1)$ games. Therefore, the total number of games which are played is
>
> $$\frac{1}{2} \times n \times 6 \times (n-1) = 3 \times n \times (n-1)$$
>
> the $\frac{1}{2}$ is included because otherwise each game is counted twice, once by each team playing. We can now try different values of n to see what works: if $n = 10$,
>
> $$3 \times n \times (n-1) = 3 \times 10 \times 9 = 270 < 396$$
>
> so n is now too small. If $n = 15$,
>
> $$3 \times n \times (n-1) = 3 \times 15 \times 14 = 630 > 396$$
>
> so n is now too big. If $n = 12$,
>
> $$3 \times n \times (n-1) = 3 \times 12 \times 11 = 396 = 396$$
>
> therefore $n = 12$.

9. A regular deck has 52 cards. There are 13 cards of each suit: Hearts, Diamonds, Spades, and Clubs. In each suit the 13 cards are numbered $2, 3, ..., 10, J, Q, K, A$. A flush consists of a group of 5 cards of the same suit. How many different flushes are there in a regular deck?

Solution: For a single suit, there are ${}_{13}C_5$ ways of choosing 5 cards. Hence, there are ${}_{13}C_5$ flushes of that suit. Since there are 4 suits, the total number of flushes in a regular deck is

$$4 \times {}_{13}C_5 = 5148$$

10. (*Fermat 2019 # 18*) How many 7-digit positive integers are made up of the digits 0 and 1 only, and are divisible by 6?

Solution: For a 7-digit positive integer to be divisible by 6, it must be even and the sum of its digits must be divisible by 3. Therefore, if it is made up of only 0's and 1's, it must end with a 0 and have either 0, 3, or 6 $1'$'s. Note the number must start with a 1 (leading 0's are not allowed) therefore it can't have 0 1's.

There is only 1 such integer with 6 1's: 1111110.

There are ${}_5C_2 = 10$ such integers which have 3 1's (since there are ${}_5C_2$ ways of picking where the two non-leading 1's go).

Therefore there are 11 such integers in total. See Fermat 2019 solutions for further details.

11. An ant starts at $(0,0)$ on the coordinate plane. The ant can repeatedly move either up one unit or to the right one unit. That is, if the ant is at (x,y) then it can either move to $(x+1,y)$ or $(x,y+1)$. How many paths could the ant take to reach $(5,5)$?

Solution: The ant will have to take 10 moves to reach $(5,5)$. We will denote up one unit as U and right one unit as R. The ant will need to do 5 U's and 5 R's. Thus, the number of paths the ant can take is the same as the number of rearrangements of 5 U's and 5 R's. Given 10 places for the moves/letters, there are ${}_{10}C_5$ ways of choosing where the U's will go and the R's will go in the remaining 5 spots. We can conclude there are

$$ {}_{10}C_5 = \frac{10!}{5! \times 5!} = 252 $$

possible paths the ant can take.

12. Find a formula for the number of diagonals in a convex polygon with n sides.
Recall: A convex polygon is a polygon such that every angle is less than 180°. A diagonal in a convex polygon is a line segment between two non-adjacent vertices (two vertices are called adjacent if the line segment between them is a side of the polygon).

Solution: Consider a convex polygon with n vertices and n sides. Any pair of vertices can be used to form a line segment which will either be a side of the polygon or a diagonal. There are ${}_nC_2$ pairs of vertices and n sides of the polygon. Hence, the number of diagonals is

$$ \begin{aligned} {}_nC_2 - n &= \frac{n!}{(n-2)! \times 2!} - n \\ &= \frac{n \times (n-1) \times (n-2)!}{2 \times (n-2)!} - n \\ &= \frac{1}{2} \times n \times (n-1) - n \\ &= \frac{1}{2} \times n \times (n-3) \end{aligned} $$